MEM6810 Engineering Systems Modeling and Simulation
工程系统建模与仿真

Theory Analysis

Lecture 1: Introduction to Simulation

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上海交通大學 SHANGHAI JIAO TONG UNIVERSITY
董浩云智能制造与服务管理研究院
CY TUNG Institute of Intelligent Manufacturing and Service Management
（中美物流研究院）
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Contents

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- Simulation is EVERYWHERE!

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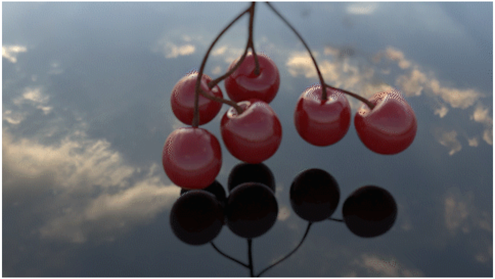

Figure: Physical Simulation of Solid-Fluid Interaction (*from* *Ruan et al. (2021)*)

What is Simulation?

Figure: Pilot Training in Boeing 787 Flat Panel Trainer (*from* Boeing)

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Figure: Airport Simulation (*by Vancouver Airport Services*)

[Video: https://www.youtube.com/watch?v=JuXwEbAvk2Q]

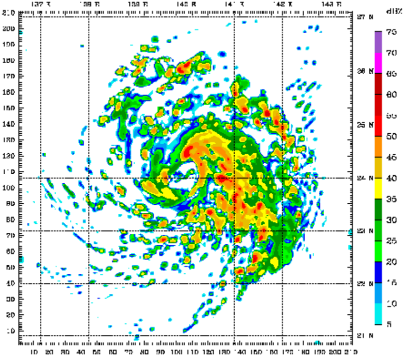

Figure: Typhoon Simulation (*image* by *Atmoz* / *CC BY 3.0*)

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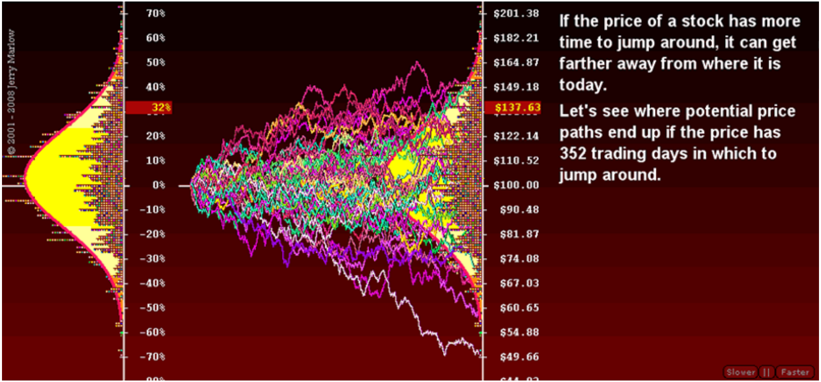

Figure: Financial Analysis

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- With simulation technique, we can easily make change and observe the effect, while keeping high fidelity.

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2. A design tool: To study systems in the design stage, before they are built.
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- Simulation is also an important type of numerical methods.

How to Do Simulation?

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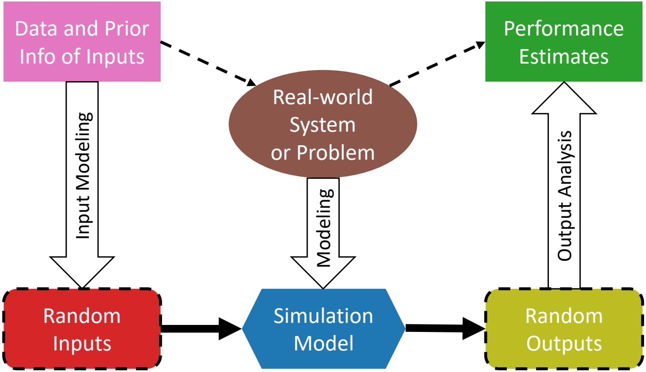

Figure: Basic Paradigm of A Simulation Study

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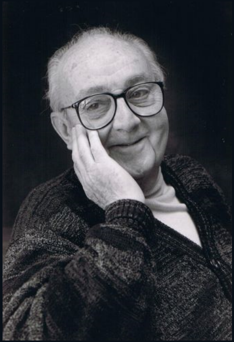
- A simulation model is a particular type of mathematical model.



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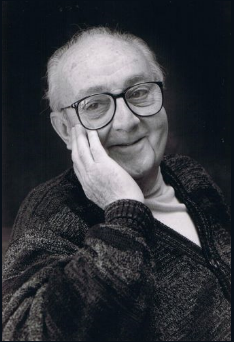

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George E. P. Box (1919.10 – 2013.03) was a British statistician, who worked in the areas of quality control, time-series analysis, design of experiments, and Bayesian inference. He has been called “one of the great statistical minds of the 20th century”.

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- Essentially, running simulation is still one type of numerical methods.
  - Real-world simulation models can be large, and such runs are usually conducted with the aid of a computer.

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   - Used much more often (uncertainty is more or less involved in a real-world system).

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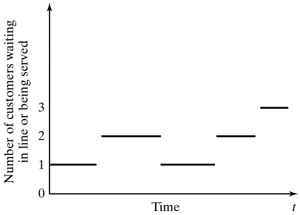

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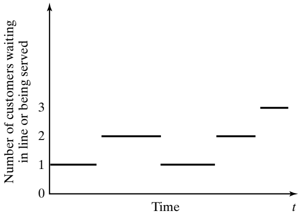

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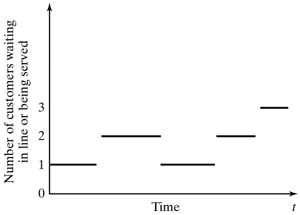

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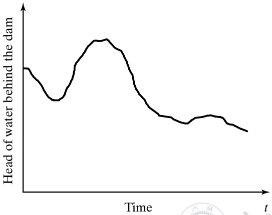

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- For most operational decision-making problems, the suitable simulation models are *dynamic*, *stochastic* and *discrete*.
  - The simulation is called Discrete-Event System Simulation (离散事件系统仿真).
  - It is the main **focus** of this course.

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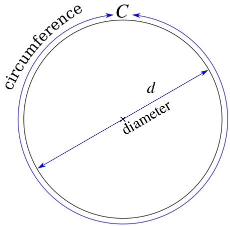

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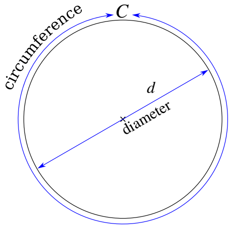

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- It was considered as a quite difficult problem in the history of mankind to find the value of π.

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  - Babylon: A clay tablet (1900–1600 BC), $\pi \approx \frac{25}{8} = 3.125$;
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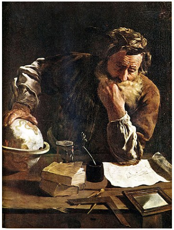

Figure: Archimedes of Syracuse (287–212 BC) (Source/Photographer)

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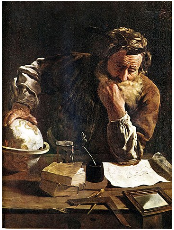

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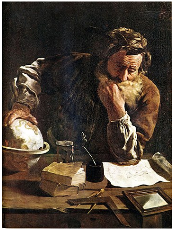

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Figure: Zu Chongzhi (祖冲之, 南北朝时期, 429–500 AD) (statue image by 三猎 / CC BY 4.0)

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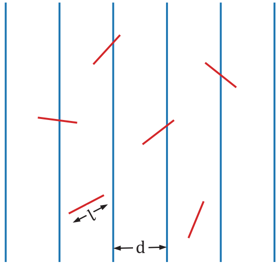

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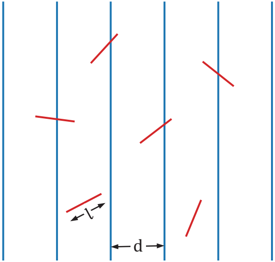

- $\mathbb{P}(\text{needle crosses a line}) = \dfrac{2l}{\pi d}$.

- If Buffon repeats the experiment for n times (i.e., drops n needles), and let h denote the number of needles crossing a line, then,

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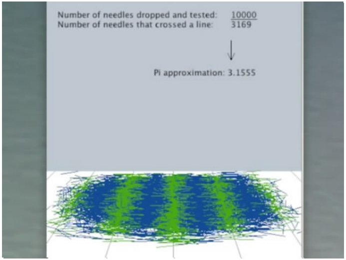

Figure: A Computer Simulation (*by Jeffrey Ventrella*)
[Video: https://www.youtube.com/watch?v=kazgQXae0Hk]

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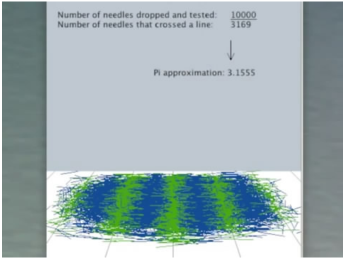

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- Try it out!

https://mste.illinois.edu/activity/buffon

http://datagenetics.com/blog/may42015/index.html

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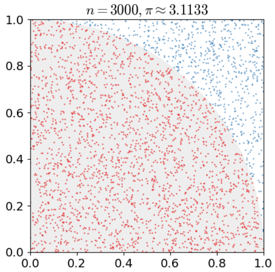

Figure: Animation (image by nicoguaro / CC BY 3.0)

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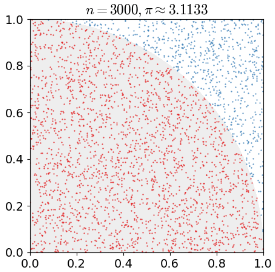

Figure: Animation (image by nicoguaro / CC BY 3.0)

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- Visit https://xiaoweiz.shinyapps.io/calPi for interaction.

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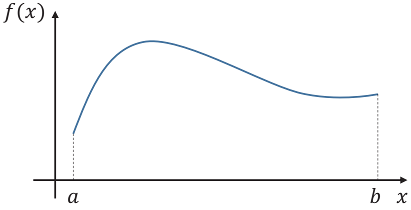

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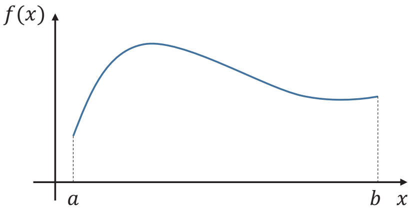

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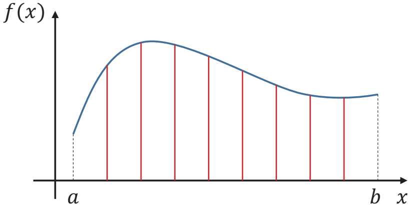

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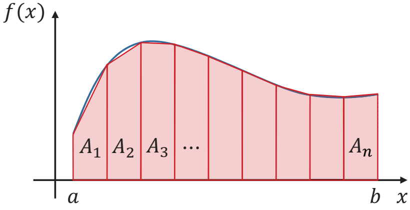

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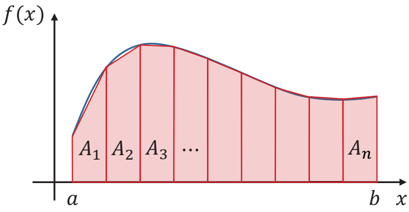

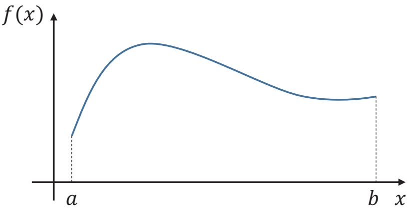

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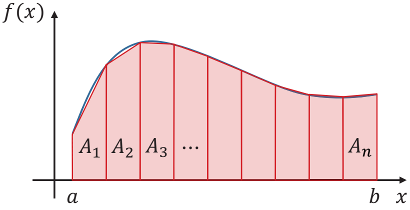

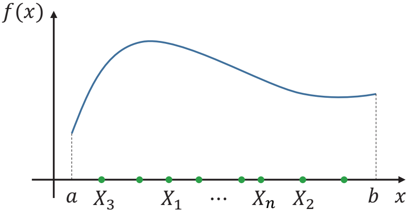

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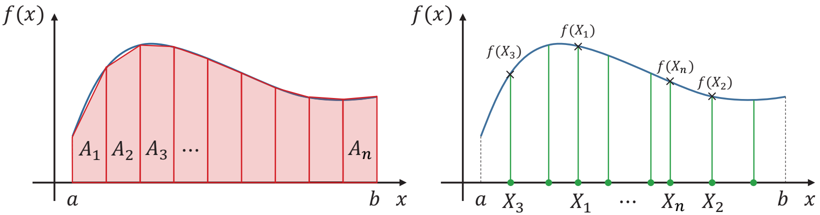

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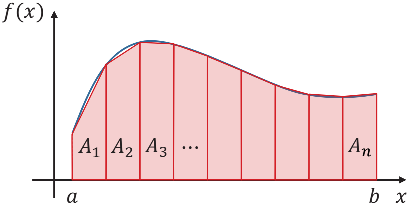

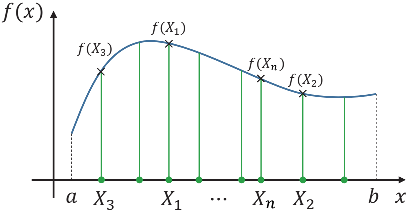

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  2. $\int_a^b f(x)\mathrm{d}x \approx \frac{b-a}{n}[f(X_1) + f(X_2) + \cdots + f(X_n)]$.
- Monte Carlo method will be much more **efficient** when the dimension is high! (E.g., $\int_{[a,b]^d} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$ for large d.)

- Recall the numerical integration problem $\int_a^b f(x)\mathrm{d}x$.

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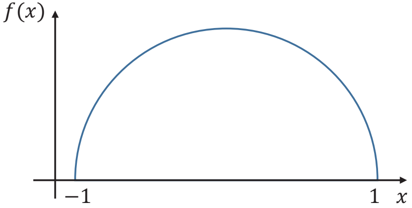
- Let $f(x) = \sqrt{1-x^2}$, $a = -1$, $b = 1$.



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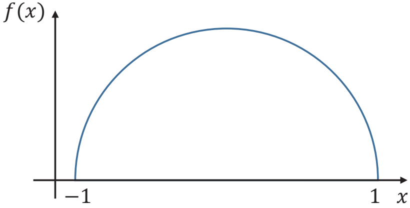

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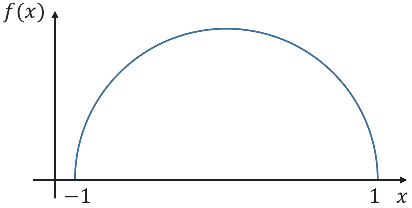

- Then, $\int_{-1}^{1} \sqrt{1-x^2}\mathrm{d}x = \pi/2$.
- So we have another way to estimate π using Monte Carlo simulation (provided we know how to compute square root).

- There is a system:
  - Two components work as active and spare, so the system fails if both components are failed.
  - Suppose the time to next component failure is random (when there is at least one functional components), which follows a known distribution, and we know how to generate it.
  - To make it simple, suppose the time to next failure is equally likely 1, 2, 3, 4, 5 or 6 days (no memory).
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- What can we say about the time to failure for this system?
- Let's run a simulation by hand!
  - Let the system **state** denote the number of functional components.
  - The **events** are the failure of a component and the completion of repair.

		Event Calendar	
Clock	System State	Next Failure	Next Repair
0	2		

		Event Calendar	
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0	2	$0 + \mathbf{5} = 5$	

		Event Calendar	
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		Event Calendar	
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0	2	$0 + \mathbf{5} = 5$	∞
5	1		

		Event Calendar	
Clock	System State	Next Failure	Next Repair
0	2	$0 + \mathbf{5} = 5$	∞
5	1		$5 + 2.5 = 7.5$

		Event Calendar	
Clock	System State	Next Failure	Next Repair
0	2	$0+\mathbf{5}=5$	∞
5	1	$5+\mathbf{3}=8$	$5+2.5=7.5$

		Event Calendar	
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7.5	2		

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7.5	2	8	∞
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14	1		$14 + 2.5 = 16.5$

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14	1	$14 + \mathbf{1} = 15$	$14 + 2.5 = 16.5$
15	$\mathbf{0}$		

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- We can observe:
  - Time to failure $= 15$
  - Average number of functional components $=$

$$\frac{1}{15-0}\left[2(5-0) + 1(7.5-5) + 2(8-7.5) + 1(10.5-8) + 2(14-10.5) + 1(15-14)\right] = \frac{24}{15}$$

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- Some questions:
  - How to deal with the randomness?
  - How to generate the time interval of component failure?

Course Outline

- Introduction to Simulation
- Elements of Probability and Statistics
- Queueing Models
- Random Variate Generation
- Input Modeling
- Verification and Validation of Simulation Models
- Output Analysis I: Single Model
- Simulation in Excel and FlexSim
- Output Analysis II: Comparison
- Output Analysis III: Optimization